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EPR Paradox, Bell's Inequalities and Entanglement: Violation of Locality?

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The standard formulation of quantum mechanics[a] states that a quantum system, unlike a classical one, can be in a superposition of states before a measurement. The mathematics on which this formalism is built implies that if two observables do not commute, the precise knowledge of one precludes the knowledge of the other. This fact and entanglement, under the assumption of *local realism*, led Einstein, Podolsky and Rosen to conclude that quantum mechanics had to be incomplete. The aim of this seminar is to explore the EPR paradox and then demonstrate that no local hidden variable can describe observations. Superdeterminism is briefly discussed.

INTRODUCTION

For centuries, the bedrock of physics rested on two highly intuitive assumptions. The first is *realism*: the idea that physical systems possess definite, objective properties independent of whether they are being observed. The second is *locality*: the information of a system is contain in its surrounding environment. Together, this framework of *local realism*[1] provided a comforting, deterministic view of the universe that reached its zenith with Einstein's theory of relativity. However, the advent of quantum mechanics threatened to dismantle this worldview. In 1935, Albert Einstein, Boris Podolsky, and Nathan Rosen published the now-famous EPR paper [1], arguing that the quantum mechanical description of reality—with its apparent "spooky action at a distance"—must be incomplete. For nearly thirty years, the EPR paradox remained trapped in the realm of philosophical interpretation. This document explores the pivotal moment that changed everything: John Bell's 1964 theorem [2]. We will examine how Bell transformed a metaphysical dispute into a rigorous, testable mathematical bound known as Bell's Inequalities. By analysing the strict mathematical constraints of classical probability, we will study the Clauser–Horne–Shimony–Holt (CHSH) inequality [3] and demonstrate how the formalism of quantum mechanics explicitly violates it. Because decades of rigorous experimentation have confirmed this quantum violation, we are forced to accept a profound truth: the universe we inhabit does not obey Local Realism. Finally, this text addresses the profound epistemological fallout of this discovery. We will explore the agonizing choices physicists must make to reconcile the math with reality—whether abandoning Locality, sacrificing Realism, or invoking the radical "nuclear option" of Superdeterminism, which preserves Local Realism only by sacrificing the fundamental scientific assumptions of statistical independence and free will.

I. EPR PARADOX (1935)

The article begins by defining two characteristics for a theory to be *successful*. It must be:

- **Correct**: In agreement with experimental results.

- **Complete**: Without worrying about all the requirements for a theory to be called complete, EPR focus on one which seems to be necessary: every element of physical reality must have a counterpart in the physical theory. So, there must be a surjection[2] ρ between *theory* and *reality*.

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[a] Also known as the Copenhagen interpretation.
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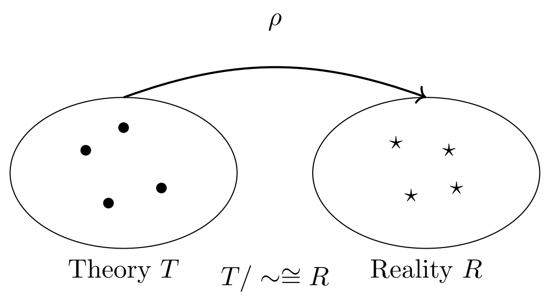
[1] Formally, local realism is the synthesis of locality and realism, mathematically requiring that the joint probability of distant measurements factorizes completely. This dictates that any observed correlations stem exclusively from a shared pre-existing state rather than non-local influence.
[2] A system can be described within different theories, then there would be more than one element associated to an element of reality. If we wish to have a bijection between theory and reality we shall get rid of redundancies and hence consider $T/\sim$ (quotient set of T by $\sim$ which is an equivalence relation whose nature will not be discussed).



This is called the condition of completeness.

Before developing the argument to arrive at a paradox, we have to ask ourselves: how are the elements of reality determined? A priori, they cannot be found under philosophical considerations. EPR gives a sufficient condition for an element to belong to physical reality: *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*. This *criterion of reality* does not intend to give a comprehensive definition of physical reality, but it should be considered as reasonable to a physicist. Otherwise, nature would not have well defined values before interacting with it which is, at least, quite surprising.
This criterion is in agreement with classical and quantum ideas. The classical case is rather evident, and for the quantum case consider a hydrogen atom in the $|nlm\rangle$ state. If we measured $\hat{H}$, $\hat{\vec{L}}^2$, $\hat{L}_z$ we would obtain with hundred percent probability the values E_n, $\hbar^2 l(l+1)$ and $\hbar m$ respectively. This means that the state $|nlm\rangle$ does correspond to an element of physical reality and hence those quantities are real.

A. Heisenberg's Uncertainty Principle

Physical observables we can measure are represented by self-adjoint operators acting on a Hilbert space. The mathematics behind this framework leads to the following property: Let $\hat{A}$ and $\hat{B}$ be two observables, their standard deviations σ_A and σ_B satisfy:

$$\sigma_A \sigma_B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle| \tag{i}$$

This inequality means that full knowledge of an observable $\hat{A}$, which does not commute with an observable $\hat{B}$, precludes the knowledge of $\hat{B}$. Let us take for example an eigenstate of the observable $\hat{p}$ in 1D $\psi(x) = e^{\frac{i}{\hbar} p_0 x}$ where p_0 is some constant value with a dimension of a momentum. It clearly verifies, by definition:

$$\hat{p}\psi(x) = \left(-i\hbar \frac{\partial}{\partial x}\right) e^{\frac{i}{\hbar} p_0 x} = p_0 e^{\frac{i}{\hbar} p_0 x} = p_0 \psi(x) \tag{ii}$$

So whenever we measure $\hat{p}$ we obtain, with certainly, the value p_0. If we decided, however, to apply $\hat{x}$,

$$\hat{x}\psi(x) = x\psi(x) \neq c^{st} \psi(x) \tag{iii}$$

we evidence that our $\psi(x)$ cannot be an eigenstate of position. Nevertheless, quantum mechanics still allows us to calculate the probability of finding our system within the interval $[a, b]$ that we note $P(a, b)$.

$$P(a, b) = \int_a^b |\psi(x)|^2 \, dx = b - a \tag{iv}$$

Since this probability only depends on the width of the interval, it follows that there is the same probability of finding our system everywhere in the 1D line so its position can be determined only by realising a measurement of $\hat{x}$ and hence collapsing the wavefunction. Let's recall the first postulate of quantum mechanics:

Postulate 1. In quantum mechanics a physical state, for example, a silver atom with a definite spin orientation, is represented by a state vector in a complex vector space. Following Dirac, we call such a vector a ket and denote it by $|\psi\rangle$. This state ket is **postulated to contain complete information about the physical state**; everything we are allowed to ask about the state is contained in the ket [4].

If we accept that the total information describing a quantum system is contained in its state vector and that the full knowledge of two quantities associated to non commuting operators is impossible, it follows that either (1) the description of reality given by the wave function in quantum mechanics is incomplete or (2) those two quantities cannot have simultaneous realities.
An obvious remark is that the negation of (1) (i.e quantum mechanics is complete) implies (2) (i.e two non commuting observables do not have simultaneous realities and that is how nature is). This might seem repetitive but it is important to understand what the Heisenberg's inequality implies. We can resume this with:

$$\neg(1) \Rightarrow (2)$$

B. Thought Experiment: Systems that Interacted in the Past

Instead of solving the Schrödinger equation, EPR proceeded to give, without any calculation, the wave function of a two-particle system they needed for the thought experiment. Although the original argument is based on the wave-function formalism, we will develop it within the bra-ket formalism for pedagogical purposes since it is totally equivalent.
Let A and B be two particles that interacted from $t = 0$ to $t = \tau$. Assume that there cannot be any interaction between them; say, for example, they are in different light cones. The particles will be in a state of anti-correlated momenta[3],

$$|\psi\rangle = |p\rangle_A \otimes |-p\rangle_B \equiv |p, -p\rangle \tag{v}$$

such that if we apply the momentum operators for A and B ($\hat{p}_A$ and $\hat{p}_B$ respectively), we obtain opposite momenta:

$$\begin{aligned} \hat{p}_A\,|\psi\rangle &= p\,|p, -p\rangle \\ \hat{p}_B\,|\psi\rangle &= -p\,|p, -p\rangle \end{aligned} \tag{vi}$$

It certainly does not mean we know what the value p is, we first need to do a measurement, but if we did, then we would know for sure what the value is for the other system, no matter how far they are. By applying $\langle x_A, x_B|$ we obtain the following wave-function:

$$\psi(x_A, x_B) = \int_{-\infty}^{+\infty} dp \left(\frac{1}{2\pi\hbar}\right) e^{\frac{i}{\hbar}px_A} e^{-\frac{i}{\hbar}px_B} e^{-\frac{i}{\hbar}px_0} \tag{vii}$$

The factor $e^{-\frac{i}{\hbar}px_0}$, where x_0 is some known value, does not change the probability because its norm is equal to one and can be seen as applying a translation $\hat{T}_A = e^{-\frac{i}{\hbar}x_0\hat{p}_A}$ to our state. This integral yields a delta "function",

$$\psi(x_A, x_B) = \delta(x_0 - (x_A - x_B)) \tag{viii}$$

where the only value with non zero probability is:

$$x_B = x_A + x_0 \tag{ix}$$

Hence, if we choose to measure the position $\hat{x}_A$, we would be able to say what the exact position of the second system is.
It is clear from (i) that we cannot know both the position and the momentum of a system simultaneously. Nevertheless, if we decided to measure the position of particle A, from (ix) we could deduce the position of particle B. Now, instead of measuring the position of A we could measure its momentum and conforming to (vi) we would know what the momentum of B is. We supposed that during the measurement there was no interaction between the two particles, we could say, as a matter of fact, that I am able to know what the position (or momentum) of particle B is by measuring $\hat{x}_A$ (resp. $\hat{p}_A$) before particle B "knows" I even did a measurement on A. Since, a priori, there cannot be superluminal interactions[4], EPR states that those two values were already well defined within B and hence, both the position and the momentum of B are real. That is, if quantum mechanics is complete and all information there is about a system

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[3] Labelling the eigenket by it's eigenvalue is purely conventional and it arises from the subjacent Lie algebra.
[4] This is what Einstein called "spooky action at a distance".

is contained in its state vector ($\neg(1)$) then two quantities associated to non commuting operators are simultaneously real[5] ($\neg(2)$). This yields:

$$\neg(1) \Rightarrow \neg(2)$$

C. Contradiction

We had two options since the very beginning, either quantum mechanics is incomplete (1) or it is complete $\neg(1)$. Assuming quantum mechanics is complete implies, on the one hand, from Heisenber's uncertainly principle, that two non commuting observables cannot be known entirely (those two quantities **are not** simultaneously real). On the other hand, from the thought experiment, without in any way perturbing system B we could predict its position and momentum (those two quantities **are** simultaneously real). Hence, saying quantum mechanics is complete ($\neg(1)$) leads to an apparent contradiction ((2) and $\neg(2)$). Then, our remaining option in order not to violate logic is:

$$\text{Quantum mechanics is incomplete} = (1)$$

This argument is totally valid under the assumption of local causality. For many years, this was not taken very seriously because asking whether a system is well defined before a measurement is, by definition, not accessible to the experimentalist.

II. BELL'S INEQUALITIES

One solution one might give to the problem just exposed is to state that $|\psi\rangle$ does not fully characterise a system, but there are some *hidden variables* (denoted λ) that represent a complete specification of the system's state, pre-programming the outcomes of any possible measurement.

A. The Bohm Formulation: Spin-1/2 Entanglement

In 1951, David Bohm simplified the continuous variables of the EPR paradox into discrete spin-1/2 systems in his textbook Quantum Theory [5]. He was simply looking for the most pedagogical and mathematically clear way to explain EPR paradox to his students. He realized that the continuous variables (position and momentum) from the original 1935 paper were mathematically cumbersome. So, he had the brilliant idea to translate the thought experiment using discrete variables (spin and Pauli algebra). Consider a source emitting a pair of entangled spin-1/2 paticles (e.g., an electron and a positron) in the singlet state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B \right) \tag{x}$$

Particle A travels to Alice, and particle B travels to Bob. Alice measures the spin along a unit vector axis $\vec{a}$, that is, she measures $\hat{S}_{\vec{a}}$. And Bob does the same with particle B along an arbitrary axis spanned by the unit vector $\vec{b}$, so he measures $\hat{S}_{\vec{b}}$.

B. Derivation of the CHSH inequality (Classical Bound)

We follow the formulation by Clauser, Horne, Shimony, and Holt (1969). Under the assumptions of Local Realism, the measurement outcomes A (for Alice) and B (for Bob) depend strictly on their local choice of measurement angle and the shared hidden variable λ.

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[5] Because we can extend the same reasoning to any two non commuting observables.

Because we are measuring spin-1/2 particles, the results are strictly dichotomous:

$$\begin{aligned} \text{Alice's outcome: } & A(\vec{a},\lambda) \in \{+1,-1\} \\ \text{Bob's outcome: } & B(\vec{b},\lambda) \in \{+1,-1\} \end{aligned} \tag{xi}$$

Crucially, Alice's outcome A does not depend on Bob's choice of angle $\vec{b}$, and vice versa. This mathematical independence is the strict expression of locality.

Let us consider two possible measurement axes for Alice ($\vec{a}$ and $\vec{a}'$) and two for Bob ($\vec{b}$ and $\vec{b}'$). We first construct an algebraic quantity, let's call it $S(\lambda)$, for a *single* pair of particles defined by a specific hidden variable state λ:

$$S(\lambda) = A(\vec{a},\lambda)B(\vec{b},\lambda) + A(\vec{a},\lambda)B(\vec{b}',\lambda) + A(\vec{a}',\lambda)B(\vec{b},\lambda) - A(\vec{a}',\lambda)B(\vec{b}',\lambda) \tag{xii}$$

By factoring out Alice's outcomes, we can rewrite this as:

$$S(\lambda) = A(\vec{a},\lambda)\Big[B(\vec{b},\lambda) + B(\vec{b}',\lambda)\Big] + A(\vec{a}',\lambda)\Big[B(\vec{b},\lambda) - B(\vec{b}',\lambda)\Big] \tag{xiii}$$

C. Justifying the Microscopic Bound

We must now rigorously evaluate the possible values of $S(\lambda)$. Because Bob's outcomes can only be $+1$ or -1, we must analyze the terms in the square brackets. There are only two logical cases:

- **Case 1: $B(\vec{b},\lambda) = B(\vec{b}',\lambda)$.**
  In this case, they have the same sign. Their sum is ± 2, and their difference is exactly 0. Therefore, the second term vanishes entirely. The equation becomes:

$$S(\lambda) = A(\vec{a},\lambda)[\pm 2] + A(\vec{a}',\lambda)[0] = \pm 2 \tag{xiv}$$

- **Case 2: $B(\vec{b},\lambda) = -B(\vec{b}',\lambda)$.**
  In this case, they have opposite signs. Their sum is exactly 0, and their difference is ± 2. Therefore, the first term vanishes entirely. The equation becomes:

$$S(\lambda) = A(\vec{a},\lambda)[0] + A(\vec{a}',\lambda)[\pm 2] = \pm 2 \tag{xv}$$

Since $A(\vec{a},\lambda)$ and $A(\vec{a}',\lambda)$ can only equal ± 1, multiplying them by ± 2 will always yield $+2$ or -2. Therefore, for *any* individual pair of particles in the universe, it is an absolute mathematical certainty that:

$$S(\lambda) \in \{+2,-2\} \implies |S(\lambda)| = 2 \tag{xvi}$$

D. Justifying the Macroscopic Bound (Expectation Value)

In a real laboratory, we cannot measure a single λ. We measure the statistical expectation value over many pairs. We define $\rho(\lambda)$ as the probability density function of the hidden variables. By the axioms of probability:

$$\rho(\lambda) \geq 0 \quad \text{and} \quad \int d\lambda\, \rho(\lambda) = 1 \tag{xvii}$$

The classical expectation value (the macroscopic correlation) for a specific pair of angles is given by the integral:

$$E(\vec{a},\vec{b}) = \int d\lambda\, \rho(\lambda) A(\vec{a},\lambda) B(\vec{b},\lambda) \tag{xviii}$$

The macroscopic CHSH quantity, $\langle S \rangle_{\text{classical}}$, is the sum of these four expectation values:

$$\langle S \rangle_{\text{classical}} = E(\vec{a},\vec{b}) + E(\vec{a},\vec{b}') + E(\vec{a}',\vec{b}) - E(\vec{a}',\vec{b}') \tag{xix}$$

Because the integral is a linear operator, we can combine this into a single integral over our microscopic quantity $S(\lambda)$ and to find the strict upper bound of this expectation value, we take the absolute value of both sides:

$$|\langle S\rangle_{\text{classical}}| = \left| \int d\lambda\, \rho(\lambda) S(\lambda) \right| \tag{xx}$$

Here, we apply the continuous triangle inequality: *The absolute value of an integral is always less than or equal to the integral of the absolute value.*

$$\left| \int d\lambda\, \rho(\lambda) S(\lambda) \right| \leq \int d\lambda\, |\rho(\lambda) S(\lambda)| \tag{xxi}$$

Since $\rho(\lambda)$ is a strictly positive probability distribution, $|\rho(\lambda)| = \rho(\lambda)$. We can pull the absolute value onto $S(\lambda)$:

$$|\langle S\rangle_{\text{classical}}| \leq \int d\lambda\, \rho(\lambda) |S(\lambda)| \tag{xxii}$$

Finally, we substitute our strict microscopic bound from Equation xvi ($|S(\lambda)| = 2$) into the integral:

$$|\langle S\rangle_{\text{classical}}| \leq \int d\lambda\, \rho(\lambda)(2) = 2 \Leftrightarrow |\langle S\rangle_{\text{classical}}| \leq 2 \tag{xxiii}$$

Any physical theory that assumes Local Realism must mathematically obey this inequality.

E. The Quantum Mechanical Violation

We now calculate the same expectation value using standard Quantum Mechanics. The observable for spin measurement along axis $\vec{a}$ is the Pauli spin operator projected onto $\vec{a}$: $\hat{\sigma} \cdot \vec{a}$. The quantum expectation value for the joint measurement $E(\vec{a}, \vec{b})$ on the singlet state is:

$$E_{QM}(\vec{a}, \vec{b}) = \langle \Psi | \, (\hat{\sigma} \cdot \vec{a}) \otimes (\hat{\sigma} \cdot \vec{b}) \, | \Psi \rangle \tag{xxiv}$$

Using the properties of Pauli matrices, this evaluates simply to the negative dot product of the measurement axes:

$$E_{QM}(\vec{a}, \vec{b}) = -\vec{a} \cdot \vec{b} = -\cos(\theta_{ab}) \tag{xxv}$$

where θ_{ab} is the angle between the detectors.

To find the maximum quantum violation of the CHSH inequality, we choose a specific geometric configuration for the detectors in a single plane:

- $\vec{a} = 0°$
- $\vec{b} = 45° \quad (\pi/4)$
- $\vec{a}' = 90° \quad (\pi/2)$
- $\vec{b}' = -45° \quad (-\pi/4)$

We compute the four required quantum correlations:

$$\begin{aligned} E_{QM}(\vec{a}, \vec{b}) &= -\cos(45°) = -\frac{1}{\sqrt{2}} \\ E_{QM}(\vec{a}, \vec{b}') &= -\cos(45°) = -\frac{1}{\sqrt{2}} \\ E_{QM}(\vec{a}', \vec{b}) &= -\cos(45°) = -\frac{1}{\sqrt{2}} \\ E_{QM}(\vec{a}', \vec{b}') &= -\cos(135°) = +\frac{1}{\sqrt{2}} \end{aligned} \tag{xxvi}$$

Substituting these into the quantum operator for S:

$$\langle S\rangle_{QM} = \left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) - \left(+\frac{1}{\sqrt{2}}\right) = -\frac{4}{\sqrt{2}} = -2\sqrt{2} \tag{xxvii}$$

Taking the absolute value, we obtain Tsirelson's bound for quantum mechanics:

$$|\langle S\rangle_{QM}| = 2\sqrt{2} \approx 2.828 \tag{xxviii}$$

The mathematical derivation yields a profound contradiction:

$$2\sqrt{2} \not\leq 2 \tag{xxix}$$

Quantum mechanics explicitly violates the CHSH inequality. Because experiments (such as those by Alain Aspect in 1982 [6]) have confirmed the value of $2\sqrt{2}$ in laboratory settings, we are forced to conclude that the assumptions leading to Equation xxiii cannot both describe our universe. Nature cannot simultaneously obey strict Einsteinian Locality and objective Counterfactual Realism. Can it?

III. SUPERDETERMINISM

Superdeterminism is defined strictly by the negation of Statistical Independence:

$$\rho(\lambda|\vec{a},\vec{b}) \neq \rho(\lambda) \tag{xxx}$$

In a superdeterministic universe, the macroscopic measurement settings $\vec{a}$ and $\vec{b}$ and the microscopic hidden variables λ are correlated. Because they share a common causal past (ultimately tracing back to the Big Bang), the laws of physics dictate that certain hidden variables will only ever be measured by certain detector alignments.

If $\rho(\lambda)$ changes depending on the variables $\vec{a}$ and $\vec{b}$, we can no longer factor it out as a constant weight across the four terms of the CHSH quantity. The derivation of the $S \leq 2$ bound mathematically collapses.

Therefore, a superdeterministic theory can reproduce the quantum correlation $E_{QM}(\vec{a},\vec{b}) = -\vec{a}\cdot\vec{b}$ and violate Bell's inequality while maintaining strict Einsteinian Locality (no faster-than-light signals) and Counterfactual Realism (definite pre-existing properties).

A. The Epistemological Crisis: Why is it Rejected?

Despite offering a mathematically valid loophole that saves Local Realism, Superdeterminism is rejected by the overwhelming majority of physicists. The rejection is not based on mathematical inconsistency, but on profound epistemological consequences: the death of the scientific method. The foundation of empirical science is the randomized control trial. To deduce causality, an experimenter must be able to isolate variables. We must assume that our choice of what to measure (whether made by human free will, a coin flip, or light from distant quasars) is statistically independent of the system we are testing. If Superdeterminism is true, we can never safely assume two variables are independent. If the experimenter's choice of setting $\vec{a}$ is correlated with the hidden state λ of the particle, then the universe is actively conspiring to present a specific result. Science becomes impossible because we can never establish true, isolated cause and effect.

CONCLUSION

Our journey today started with a simple, intuitive desire: to describe a universe that makes classical sense. We wanted a universe ruled by **Locality** (where things only touch their immediate neighbours) and **Reality** (quantities are well-defined before a measurement). But Quantum Mechanics forced us to look deeper, and the EPR paradox laid the ultimate trap. By trying to prove that the quantum map was incomplete, EPR inadvertently sparked one the greatest philosophical crisis in modern physics. This single inequality proved that the classical dream of **Local Realism** is dead. Thus, we must make a choice between: abandon Locality, abandon Realism or abandon Statistical Independence (Superdeterminism).

LICENSING

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